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COMMENT

Rational solutions of the classical Boussinesq hierarchy

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Abstract. The classical Boussinesq system and its associated higher-order flows in the bilinear form are considered. An infinite family of homogeneous rational solutions of this hierarchy is verified by the use of the Wronskian technique developed by Freeman and Nimmo.

1. Introduction

The classical Boussinesq equation [1-6]

$$u_t = w_x + uu_x \qquad w_t = u_{xxx} + (uw)_x \tag{1}$$

describing dispersive water waves, is a $pq = 0$ reduction of the modified Kadomtsev-Petviashvili equation [7] and belongs to the reduced two-component Kadomtsev-Petviashvili hierarchy [8]. An infinite family of homogeneous rational solutions of this equation has been constructed in terms of Wronskians of certain basic polynomials which arise in group representation theory and verified by the use of the Wronskian method [9-11].

The Wronskian method developed by Freeman and Nimmo [12-14] is a simple and useful way to investigate solutions of the soliton equations in bilinear form. In this method, the proof that the bilinear soliton solutions should satisfy the bilinear equations can be obtained by the use of the Laplace expansion theorem of the matrix algebra. In this comment, we apply this method to the classical Boussinesq hierarchy [15]

$$\begin{pmatrix} u_t \\ w_t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(u + u_x\partial_x^{-1}) & 1 \\ \partial_x^2 + w + \frac{1}{2}w_x\partial_x^{-1} & \frac{1}{2}u \end{pmatrix}^N \begin{pmatrix} u_x \\ w_x \end{pmatrix} \qquad N = 1, 2, \ldots. \tag{2}$$

2. Rational solutions: the Wronskian technique

By introducing an infinite number of time variables $x = t_1, t_2, t_3, \ldots$ and considering u, w as functions of $t = (t_1, t_2, t_3, \ldots)$, we have an equivalent form of (2)

$$\begin{pmatrix} u_{t_{N+1}} \\ w_{t_{N+1}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(u + u_x\partial_x^{-1}) & 1 \\ \partial_x^2 + w + \frac{1}{2}w_x\partial_x^{-1} & \frac{1}{2}u \end{pmatrix} \begin{pmatrix} u_{t_N} \\ w_{t_N} \end{pmatrix} \qquad N = 1, 2, \ldots. \tag{3}$$

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In the following, we assume that u and w vanish rapidly as $|x|\to\infty$. Through the dependent variable transform

$$u=2(\log(\sigma/\tau))_x \qquad w=2(\log\sigma\tau)_{xx}$$

equation (3) is transformed into the bilinear form [16-18]

$$(D_{t_{N+1}}-D_{t_1}D_{t_N})\sigma\cdot\tau=0 \qquad (2D_{t_1}D_{t_{N+1}}-D_{t_2}D_{t_N}-D_{t_1}^2D_{t_N})\sigma\cdot\tau=0$$

$$N=1,2,\ldots. \tag{4}$$

We introduce $q_r(t)$ by the generating function

$$\exp(kt_1+k^2t_2+k^3t_3+\ldots)=\sum_{r=-\infty}^{\infty}k^rq_r(t). \tag{5}$$

It follows directly from (5) that

$$\partial_{t_1}^{l_1}\partial_{t_2}^{l_2}\ldots\partial_{t_k}^{l_k}\ldots q_r(t)=q_\rho(t)$$

where $\rho=r-l_1-2l_2-\ldots-kl_k-\ldots$ and $q_r(t)=0$ for $r<0$.

For simplicity, we introduce an abbreviated notation for the Wronskian of the functions $q_{r_1}, q_{r_2},\ldots,q_{r_k}$ in t_1:

$$(r_1,r_2,\ldots,r_k)=\begin{vmatrix} q_{r_1} & q_{r_2} & \cdots & q_{r_k}\\ \partial_{t_1}q_{r_1} & \partial_{t_1}q_{r_2} & \cdots & \partial_{t_1}q_{r_k}\\ \vdots & \vdots & \ddots & \vdots\\ \partial_{t_1}^{k-1}q_{r_1} & \partial_{t_1}^{k-1}q_{r_2} & \cdots & \partial_{t_1}^{k-1}q_{r_k}\end{vmatrix}$$

$$=\begin{vmatrix} q_{r_1} & q_{r_2} & \cdots & q_{r_k}\\ q_{r_1-1} & q_{r_2-1} & \cdots & q_{r_k-1}\\ \vdots & \vdots & \ddots & \vdots\\ q_{r_1-k+1} & q_{r_2-k+1} & \cdots & q_{r_k-k+1}\end{vmatrix}.$$

This notation is similar to the one used by Freeman and Nimmo [12] and is more relevant to our problem.

The purpose of this comment is to prove the following theorem.

Theorem. For all integers m and n $(m<n)$, the pair

$$\sigma=(n,n-1,\ldots,m) \qquad \tau=(n,n-1,\ldots,m+1)$$

satisfies the hierarchy (4).

Proof. Substitution of the expressions for σ and τ into the first part of (4) gives

$$\begin{aligned}(D_{t_{N+1}}&-D_{t_1}D_{t_N})\sigma\cdot\tau\\ &=(\sigma_{t_{N+1}}-\sigma_{t_1t_N})\tau+\sigma_{t_N}\tau_{t_1}-\sigma(\tau_{t_{N+1}}+\tau_{t_1t_N})+\sigma_{t_1}\tau_{t_N}\\ &=\sum_{k=m+1}^{n}(-1)^{k-m}(n,\ldots,\tilde{k},\ldots,m,k-N-1)(n,\ldots,m+1)\\ &\quad-\sum_{k=m+1}^{n}(-1)^{k-m}(n,\ldots,\tilde{k},\ldots,m+1,m-1,k-N)(n,\ldots,m+1)\\ &\quad+\sum_{k=m}^{n}(-1)^{k-m}(n,\ldots,\tilde{k},\ldots,m,k-N)(n,\ldots,m+2,m)\end{aligned}$$

$$
\begin{aligned}
&-(n,\dots,m)(n,\dots,m+2,m-N)\\
&+\sum_{k=m+1}^{n}(-1)^{k-m}(n,\dots,m)(n,\dots,\tilde{k},\dots,m+1,k-N-1)\\
&+\sum_{k=m+2}^{n}(-1)^{k-m}(n,\dots,m)(n,\dots,\tilde{k},\dots,m+2,m,k-N)\\
&-\sum_{k=m+1}^{n}(-1)^{k-m}(n,\dots,m+1,m-1)(n,\dots,\tilde{k},\dots,m+1,k-N)
\end{aligned}
$$

where $\tilde{k}$ means that the function q_k is cancelled in the Wronskian. Using the Laplace expansion theorem, we have

$$
\begin{aligned}
(n,\dots,\tilde{k},\dots,&m+1,m-1,k-N)(n,\dots,m+1)\\
&+(n,\dots,m+1,m-1)(n,\dots,\tilde{k},\dots,m+1,k-N)\\
&-(n,\dots,m+1,k-N)(n,\dots,\tilde{k},\dots,m+1,m-1)\\
&=\left|\begin{array}{c|c|c|c|c}
n,\dots,\tilde{k},\dots,m+1 & 0 & k & m-1 & k-N\\ \hline
0 & n,\dots,\tilde{k},\dots,m+1 & k & m-1 & k-N
\end{array}\right|=0.
\end{aligned}
$$

Here and below in the $(2n-2m+1)\times(2n-2m+1)$ determinants, l above the horizontal line denotes $(q_l,\dots,q_{l-n+m})^{\mathrm{T}}$ and l below the horizontal line denotes $(q_l,\dots,q_{l-n+m+1})^{\mathrm{T}}$. Also,

$$
\begin{aligned}
(n,\dots,\tilde{k},\dots,m,k-N)&(n,\dots,m+2,m)+(n,\dots,m)(n,\dots,\tilde{k},\dots,m+2,m,k-N)\\
&-(n,\dots,m+2,m,k-N)(n,\dots,\tilde{k},\dots,m)\\
&=\left|\begin{array}{c|c|c|c|c}
n,\dots,\tilde{k},\dots,m+2,m & 0 & k & m+1 & k-N\\ \hline
0 & n,\dots,\tilde{k},\dots,m+2,m & k & m+1 & k-N
\end{array}\right|=0.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(D_{t_{N+1}}-D_{t_1}D_{t_N})\sigma\cdot\tau
&=\sum_{k=m+1}^{n}(-1)^{k-m}(n,\dots,\tilde{k},\dots,m,k-N-1)(n,\dots,m+1)\\
&\quad-\sum_{k=m+1}^{n}(-1)^{k-m}(n,\dots,m+1,k-N)(n,\dots,\tilde{k},\dots,m+1,m-1)\\
&\quad-(n,\dots,m)(n,\dots,m+2,m-N)\\
&\quad+\sum_{k=m+1}^{n}(-1)^{k-m}(n,\dots,m+2,m,k-N)(n,\dots,\tilde{k},\dots,m)\\
&\quad+\sum_{k=m+1}^{n}(-1)^{k-m}(n,\dots,m)(n,\dots,\tilde{k},\dots,m+1,k-N-1)\\
&\quad+(n,\dots,m+1,m-N)(n,\dots,m+2,m)\\
&=-(n,\dots,m+2,m,m-N)(n,\dots,m+1)\\
&\quad-(n,\dots,m)(n,\dots,m+2,m-N)\\
&\quad+(n,\dots,m+1,m-N)(n,\dots,m+2,m)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=m+1}^{n-1}(-1)^{k-m}[-(n,\ldots,\widetilde{k+1},\ldots,m,k-N)(n,\ldots,m+1)\\
&-(n,\ldots,m+1,k-N)(n,\ldots,\tilde{k},\ldots,m+1,m-1)\\
&+(n,\ldots,m+2,m,k-N)(n,\ldots,\tilde{k},\ldots,m)\\
&+(n,\ldots,m)(n,\ldots,\tilde{k},\ldots,m+1,k-N-1)]\\
&+(-1)^{n-m}[-(n,\ldots,m+1,n-N)(n-1,\ldots,m+1,m-1)\\
&+(n,\ldots,m)(n-1,\ldots,m+1,n-N-1)\\
&+(n,\ldots,m+2,m,n-N)(n-1,\ldots,m)]
\end{aligned}
$$

$$
\begin{aligned}
&=-\left|\begin{array}{c|c|c|c|c}
n,\ldots,m+2 & 0 & m+1 & m & m-N\\ \hline
0 & n,\ldots,m+2 & m+1 & m & m-N
\end{array}\right|\\
&+\sum_{k=m+1}^{n-1}(-1)^{n-m}\left|\begin{array}{c|c|c|c|c|c}
n,\ldots,\widetilde{k+1},\ldots,m+2 & 0 & k+1 & m+1 & m & k-N\\ \hline
0 & n,\ldots,\tilde{k},\ldots,m+1 & k & m & m-1 & k-N-1
\end{array}\right|+(-1)^{n-m}\\
&\times\left|\begin{array}{c|c|c|c|c}
n,\ldots,m+2 & 0 & m+1 & m & n-N\\ \hline
0 & n-1,\ldots,m+1 & m & m-1 & n-N-1
\end{array}\right|=0.
\end{aligned}
$$

Thus the first part of the theorem has been proved. To prove the second part of the theorem, first we notice an identity

$$
\begin{aligned}
&(2D_{t_1}D_{t_{N+1}}-D_{t_2}D_{t_N}-D_{t_1}^2D_{t_N})\sigma\cdot\tau-2[(D_{t_{N+1}}-D_{t_1}D_{t_N})\sigma\cdot\tau]_{t_1}+[(D_{t_2}-D_{t_1}^2)\sigma\cdot\tau]_{t_N}\\
&\qquad=4(\sigma_{t_{N+1}}-\sigma_{t_1t_N})\tau_{t_1}+4\sigma_{t_N}\tau_{t_1t_1}+2(\sigma_{t_1t_1}-\sigma_{t_2})\tau_{t_N}-2\sigma(2\tau_{t_1t_{N+1}}+\tau_{t_1t_1t_N}-\tau_{t_2t_N}).
\end{aligned}
$$

So it is sufficient to prove that the right-hand side of this identity equals zero.

Substitution of the expressions for σ and τ into the second part of (4) gives

$$
\begin{aligned}
&(\sigma_{t_{N+1}}-\sigma_{t_1t_N})\tau_{t_1}+\sigma_{t_N}\tau_{t_1t_1}+\tfrac{1}{2}(\sigma_{t_1t_1}-\sigma_{t_2})\tau_{t_N}-\sigma[\tau_{t_1t_{N+1}}+\tfrac{1}{2}(\tau_{t_1t_1t_N}-\tau_{t_2t_N})]\\
&\qquad=\sum_{k=m+1}^{n}(-1)^{k-m}[(n,\ldots,\tilde{k},\ldots,m,k-N-1)\\
&\qquad-(n,\ldots,\tilde{k},\ldots,m+1,m-1,k-N)](n,\ldots,m+2,m)\\
&\qquad+\sum_{k=m}^{n}(n,\ldots,\tilde{k},\ldots,m,k-N)[(n,\ldots,m+2,m-1)\\
&\qquad+(n,\ldots,m+3,m+1,m)]\\
&\qquad-\sum_{k=m+1}^{n}(-1)^{k-m}(n,\ldots,m+2,m,m-1)(n,\ldots,\tilde{k},\ldots,m+1,k-N)\\
&\qquad-(n,\ldots,m)(n,\ldots,m+2,m-N-1)+\sum_{k=m+2}^{n}(-1)^{k-m}(n,\ldots,m)\\
&\qquad\times(n,\ldots,\tilde{k},\ldots,m+2,m,k-N-1)-(n,\ldots,m)\\
&\qquad\times(n,\ldots,m+3,m+1,m-N)\\
&\qquad+(n,\ldots,m)(n,\ldots,m+3,m,m-N+1)\\
&\qquad+\sum_{k=m+3}^{n}(-1)^{k-m}(n,\ldots,m)(n,\ldots,\tilde{k},\ldots,m+3,m+1,m,k-N).
\end{aligned}
$$

Using the Laplace expansion theorem, we have

$$
\begin{aligned}
&(n,\ldots,\tilde{k},\ldots,m,k-N)(n,\ldots,m+3,m+1,m)\\
&\qquad+(n,\ldots,m)(n,\ldots,\tilde{k},\ldots,m+3,m+1,m,k-N)\\
&\qquad-(n,\ldots,m+3,m+1,m,k-N)(n,\ldots,\tilde{k},\ldots,m)\\
&\quad=\left|\begin{array}{c|c|c|c}
n,\ldots,\tilde{k},\ldots,m+3,m+1,m & & & \\ \hline
0 & & & \\ \hline
0 & k & m+2 & k-N \\ \hline
n,\ldots,\tilde{k},\ldots,m+3,m+1,m & k & m+2 & k-N
\end{array}\right|=0.
\end{aligned}
$$

Also

$$
\begin{aligned}
&-(n,\ldots,\tilde{k},\ldots,m+1,m-1,k-N)(n,\ldots,m+2,m)\\
&\qquad+(n,\ldots,\tilde{k},\ldots,m,k-N)(n,\ldots,m+2,m-1)\\
&\qquad-(n,\ldots,m+2,m,m-1)(n,\ldots,\tilde{k},\ldots,m+1,k-N)\\
&\qquad+(n,\ldots,m+1,k-N)(n,\ldots,\tilde{k},\ldots,m+2,m,m-1)\\
&\quad=(-1)^{n-k}\left|\begin{array}{c|c|c|c|c|c}
n,\ldots,\tilde{k},\ldots,m+2 & & & & & \\ \hline
0 & & & & & \\ \hline
0 & k & m+1 & m & m-1 & k-N \\ \hline
n,\ldots,\tilde{k},\ldots,m+2 & k & k-N & m & m-1 & m+1
\end{array}\right|\\
&\qquad-(-1)^{k-m}\left|\begin{array}{c|c|c|c|c}
n,\ldots,\tilde{k},\ldots,m+1 & 0 & k & m & m-1 \\ \hline
0 & n,\ldots,\tilde{k},\ldots,m+1 & k & m & m-1
\end{array}\right|\\
&\qquad-(-1)^{k-m}\left|\begin{array}{c|c|c|c}
n,\ldots,\tilde{k},\ldots,m+2,k-N & & & \\ \hline
0 & & & \\ \hline
0 & k & m & m-1 \\ \hline
n,\ldots,\tilde{k},\ldots,m+2,k-N & k & m & m-1
\end{array}\right|=0.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&(\sigma_{t_{N+1}}-\sigma_{t_1t_N})\tau_{t_1}+\sigma_{t_N}\tau_{t_1t_1}+\tfrac{1}{2}(\sigma_{t_1t_1}-\sigma_{t_2})\tau_{t_N}-\sigma[\tau_{t_1t_{N+1}}+\tfrac{1}{2}(\tau_{t_1t_1t_N}-\tau_{t_2t_N})]\\
&\quad=\sum_{k=m+2}^{n-1}(-1)^{k-m}[-(n,\ldots,\widetilde{k+1},\ldots,m,k-N)(n,\ldots,m+2,m)\\
&\qquad-(n,\ldots,m+1,k-N)(n,\ldots,\tilde{k},\ldots,m+2,m,m-1)\\
&\qquad+(n,\ldots,m+3,m+1,m,k-N)(n,\ldots,\tilde{k},\ldots,m)\\
&\qquad+(n,\ldots,m)(n,\ldots,\tilde{k},\ldots,m+2,m,k-N-1)]\\
&\qquad-(n,\ldots,m+2,m,m-N)(n,\ldots,m+2,m)\\
&\qquad+(n,\ldots,m+3,m+1,m,m-N+1)(n,\ldots,m+2,m)\\
&\qquad+(n,\ldots,m+2,m-1,m-N+1)(n,\ldots,m+2,m)\\
&\qquad+(n,\ldots,m+1,m-N)(n,\ldots,m+2,m-1)\\
&\qquad-(n,\ldots,m+2,m,m-N+1)(n,\ldots,m+2,m-1)
\end{aligned}
$$

$$
\begin{aligned}
&+(n,\ldots,m+2,m,m-1)(n,\ldots,m+2,m-N+1)\\
&+(n,\ldots,m+1,m-N)(n,\ldots,m+3,m+1,m)\\
&-(n,\ldots,m+2,m,m-N+1)(n,\ldots,m+3,m+1,m)\\
&-(n,\ldots,m)(n,\ldots,m+2,m-N-1)\\
&-(n,\ldots,m)(n,\ldots,m+3,m+1,m-N)\\
&+(n,\ldots,m)(n,\ldots,m+3,m,m-N+1)\\
&+(-1)^{n-m}[-(n,\ldots,m+1,n-N)(n-1,\ldots,m+2,m,m-1)\\
&+(n,\ldots,m+3,m+1,m,n-N)(n-1,\ldots,m)\\
&+(n,\ldots,m)(n-1,\ldots,m+2,m,m-N-1)]
\end{aligned}
$$

$$
\begin{aligned}
=&-\sum_{k=m+2}^{n-1}(-1)^{n-m}\left|\begin{array}{c:c:c:c:c:c}
n,\ldots,\widetilde{k+1},\ldots,m+3,m+1 & 0 & k+1 & m+2 & m & k-N\\ \hdashline
0 & n,\ldots,\tilde{k},\ldots,m+2,m & k & m+1 & m-1 & k-N-1
\end{array}\right|\\
&+(-1)^{n-m}\left|\begin{array}{c:c:c:c:c:c}
n,\ldots,m+3 & 0 & m+2 & m+1 & m & m-N\\ \hdashline
0 & n,\ldots,m+2 & m+1 & m & m-1 & m-N-1
\end{array}\right|\\
&+\left|\begin{array}{c:c:c:c:c}
n,\ldots,m+3,m+1 & 0 & m+2 & m & m-N\\ \hdashline
0 & n,\ldots,m+3,m+1 & m+2 & m & m-N
\end{array}\right|\\
&+\left|\begin{array}{c:c:c:c:c}
n,\ldots,m+2 & 0 & m & m-1 & m-N+1\\ \hdashline
0 & n,\ldots,m+2 & m & m-1 & m-N+1
\end{array}\right|\\
&+\left|\begin{array}{c:c:c:c:c}
n,\ldots,m+3,m & 0 & m+2 & m+1 & m-N+1\\ \hdashline
0 & n,\ldots,m+3,m & m+2 & m+1 & m-N+1
\end{array}\right|\\
&-(-1)^{n-m}\left|\begin{array}{c:c:c:c:c}
n,\ldots,m+3,m+1 & 0 & m+2 & m & n-N\\ \hdashline
0 & n-1,\ldots,m+2,m & m+1 & m-1 & n-N-1
\end{array}\right|=0.
\end{aligned}
$$

Thus the second part of the theorem has also been proved. □

3. Remark

We have found that many famous soliton equations and their higher-order flows have simple bilinear forms. These bilinear forms are more concise than they used to be and easier to treat. For example, we have proved the Bäcklund transformations and the nonlinear superposition formulae of the KdV and MKdV hierarchies by using Hirota's method [18]. One of our present interests is to use the Wronskian method to study not a single system but the whole hierarchy. This comment is the first attempt in this direction. We shall report other results elsewhere.

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